The Vector Field Problem for Projective Stiefel Manifolds

Július Korbaš Peter Zvengrowski

Dedicado afectuosamente a la memoria de Guillermo Moreno – un entusiasta de las álgebras de Cayley-Dickson y sus aplicaciones en topología

Abstract

Results for the vector field problem on projective Stiefel manifolds $X_{n,r} \cong O(n)/(O(n-r) \times \mathbb{Z}_2)$, $2 \leq r < n$, are derived here; $X_{n,1}$ is $(n-1)$-dimensional real projective space, for which these results are classical. In particular, $\operatorname{span}(X_{n,r})$ for $r = 2, 3, 4$, for suitable (infinitely many) values of n is calculated. If $r = 2$ and n is odd, then additional difficulties present themselves, and one approach to dealing with this case using the Browder-Dupont invariant is discussed. Furthermore, when $n = 8m-1$, by using an explicit version of the Hurwitz-Radon multiplications, we improve the lower bound for $\operatorname{span}(X_{n,2})$ to $\operatorname{span}(S^n)$. Two general results and some conjectures on the span of $X_{n,r}$ are also presented.

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1 Introduction

The span of a finite dimensional real vector bundle α over a space X, denoted $\operatorname{span}(\alpha)$, is k if α admits k, but no more than k, everywhere linearly independent cross-sections. If X is paracompact, then $\operatorname{span}(\alpha) \geq k$ means that $\alpha \approx \eta \oplus k\varepsilon$ for some vector bundle η; here and in the sequel ε is the trivial line bundle and $k\varepsilon$ denotes the k-fold Whitney sum of ε with itself.

For a q-dimensional smooth connected manifold M^q, one defines its span to be $\operatorname{span}(M^q) := \operatorname{span}(TM^q)$, where TM is the tangent bundle of M. The manifold M^q is parallelizable if its span is q. The problem of determining the number $\operatorname{span}(M)$ is referred to as the vector field problem on M (further information can be found in [16], [18], [19], [20], [34]).

Besides the span of a manifold one can consider its stable span ([16], [18], [19], [20]):

$$\operatorname{span}^0(M) := \operatorname{span}(TM \oplus \varepsilon) - 1. \tag{1}$$

We remark that the stable span of a given smooth closed manifold M is interesting even if one is not able to find its span, in the context of fold maps (a smooth map $f : M^q \to N^p$ with $q \geq p$ is a fold map if all of its singular points are of fold type; a singular point $x \in M$ is of fold type if for some local coordinates around x and $f(x)$ one can write f as the assignment $(x_1, \ldots, x_q) \mapsto (x_1, \ldots, x_{p-1}, \pm x_p^2 \pm x_{p+1}^2 \pm \cdots \pm x_q^2)$; in particular if $N^p = \mathbb{R}$, then a fold map is a Morse function on M). By Y. Ando (cf. [30]), if $\dim(M) = q$ and $\operatorname{span}^0(M) \geq p - 1$ for some p such that $q \geq p \geq 2$, then there exists a fold map $M \to \mathbb{R}^p$. In addition to this, as proved by O. Saeki ([29]), if $q - p$ is even and there exists a fold map $M \to \mathbb{R}^p$, then $\operatorname{span}^0(M) \geq p - 1$.

Stably parallelizable manifolds (known also as π-manifolds) are those for which the stable span is the same as the dimension; the Bredon-Kosinski theorem ([10]) effectively determines the span of such manifolds. The projective Stiefel manifolds $X_{n,r}$, $1 \leq r < n$, obtained from the ordinary Stiefel manifolds $V_{n,r}$ of orthonormal r-frames in $\mathbb{R}^n$ by identifying $(v_1, \ldots, v_r) \in V_{n,r}$ with $(-v_1, \ldots, -v_r)$, form a family of closed, connected, smooth manifolds, among which relatively few are stably parallelizable (see [5]). Note that $X_{n,1} = P^{n-1}$, $(n-1)$-dimensional real projective space, for which the span question was solved by Adams [1]. The study of $X_{n,r}$ for $r > 1$ was inaugurated by P. Baum and W. Browder [8] and S. Gitler and D. Handel [11] in the 1960's, and has been a subject of ongoing interest since then ([4], [5], [6], [7], [31], [32], [35], [36] etc.).

This paper (mentioned as a "later paper" in [31]), in combination with [31], is an attempt to present the current state of knowledge concerning the span question for the projective Stiefel manifolds $X_{n,r}$, $r \geq 2$. This question is related to the same problem for other important spaces. For instance, for the flag manifold $O(n)/((O(1))^r \times O(n-r))$ we have that $\operatorname{span}(X_{n,r}) \geq \operatorname{span}(O(n)/((O(1))^r \times O(n-r))$. We note that the information available on the span of the above mentioned flag manifold is in general quite weak (see [18]; an exception: for $r = 2$, quite a bit is known; see [3], [12], [15]).

In the sequel, the number $\dim(X_{n,r}) = nr - \binom{r+1}{2}$ will be denoted by $d_{n,r}$; we shall write just d instead of $d_{n,r}$ when (n, r) is clear from the context. For the tangent bundle we have ([22], [38])

$$\tau_{n,r} := TX_{n,r} \approx r\xi_{n,r} \otimes \beta_{n,r} \ \oplus \ \binom{r}{2}\varepsilon, \tag{2}$$

and stably

$$\tau_{n,r} \oplus \binom{r+1}{2}\varepsilon \approx nr\xi_{n,r}, \tag{3}$$

where $\xi_{n,r}$ (sometimes denoted just by ξ) is the line bundle associated to the obvious double covering $V_{n,r} \longrightarrow X_{n,r}$, and $\beta_{n,r}$ is the "orthogonal complement" bundle characterized by $r\xi_{n,r} \oplus \beta_{n,r} \approx n\varepsilon$. Note that $\xi_{n,1} = \xi_{n-1}$, the familiar Hopf line bundle over $X_{n,1} = P^{n-1}$.

Of course, (stable) $\operatorname{span}(X_{n,r})$ can be at least i only if the Stiefel-Whitney classes $w_{d-i+j}(\tau_{n,r}) := w_{d-i+j}(X_{n,r})$, $j \geq 1$, vanish. The isomorphism (3) implies that

$$w(X_{n,r}) = (1 + w_1(\xi_{n,r}))^{nr}. \tag{4}$$

For deciding whether or not $w_i(X_{n,r})$ vanishes, it is also necessary to know the $\mathbb{Z}_2$-cohomology ring of $X_{n,r}$. By [11, 1.6],

$$H^*(X_{n,r}; \mathbb{Z}_2) = \mathbb{Z}_2[y]/(y^N) \otimes V(y_{n-r}, \ldots, \hat{y}_{N-1}, \ldots, y_{n-1}), \tag{5}$$

where $y = w_1(\xi_{n,r})$, $N = \min\{j; j \geq n - r + 1, \ \binom{n}{j} \equiv 1 \pmod 2\}$, and

$$V(y_{n-r}, \ldots, \hat{y}_{N-1}, \ldots, y_{n-1})$$

is the $\mathbb{Z}_2$-vector space having the monomials $\prod_{i=n-r}^{n-1} y_i^{t_i}$ with $i \neq N - 1$ and $t_i \in \{0, 1\}$ as $\mathbb{Z}_2$-basis. For later use, we note that (5) immediately implies the formula for the mod 2 Poincaré polynomial:

$$P_t(X_{n,r}; \mathbb{Z}_2) = \frac{(1 + t + \cdots + t^{N-1})(1 + t^{n-r}) \cdots (1 + t^{n-1})}{1 + t^{N-1}}.$$

Additionally, (5) determines all cup products in $H^*(X_{n,r}; \mathbb{Z}_2)$ except for y_i^2, which can be found using [4] since $y_i^2 = \mathrm{Sq}^i(y_i)$. Correcting misprints of [4], we reproduce the formulae for Steenrod squares here (these formulae were also published in [39]). Let $t := \nu_2(N)$ denote the exponent of the largest power of 2 dividing N. Then one has

$$\mathrm{Sq}^i(y_{q-1}) = \sum_{k=0}^{i} A_k y^k y_{q+i-1-k} + \sum_{0\le k<j\le i} B_{k,j} y^{q+k+i-N-j} y_{N+j-k-1} + \epsilon y^{q+i-1},$$

where

$$\epsilon = \begin{cases} \binom{n}{q+2^{t-1}-N}\binom{q+2^{t-1}-N}{i-1} & \text{if } t \ge 3, \\ 0 & \text{if } t < 3, \end{cases}$$

$$A_k = A(q,i,k) = \binom{q-1-k}{q-1-i}\binom{n}{k},$$

$$B_{k,j} = B(q,i,k,j) = \binom{n}{q}\binom{N-1-k}{N-1-j}\binom{q-N}{i-j}\binom{n}{k}.$$

Apart from $X_{12,8}$, the parallelizability question for $X_{n,r}$, $r \ge 2$, is settled in [5] and in [6]. In addition to this, a complete solution to the vector field problem on $X_{n,r}$ is known for some families of (n,r). More precisely, for $X_{n,1} = P^{n-1}$, one has, as a consequence of the solution of the vector field problem for spheres [1], that $\mathrm{span}(X_{n,1}) = \mathrm{span}(S^{n-1}) = \rho(n) - 1$, where $\rho(n) = 2^c + 8d$ for n expressed as $(2a+1)2^{c+4d}$, $a,\, d \ge 0,\ 0 \le c \le 3$. One calls $\rho(n)$ the Hurwitz-Radon number of n, and this will also be useful later in this work. At one extreme, for r close to n, it has long been known ([5]) that $X_{n,n-1}$ and $X_{2m,2m-2}$ (with m arbitrary) are parallelizable. Around 1998, Zvengrowski [39] has determined $\mathrm{span}(X_{2m+1,2m-1})$ and $\mathrm{span}(X_{n,n-3})$; thus $\mathrm{span}(X_{n,n-j})$ is also known for $j \le 3$. Our aim in this paper is to study the other extreme, r close to 2, and calculate the span of $X_{n,r}$ for some families with $r = 2$, 3, 4, as well as to prove some general results on $\mathrm{span}(X_{n,r})$.

From the formula (2), one immediately has that $\mathrm{span}(X_{n,r}) \ge 1$ when $r \ge 2$. In [18] and [19] we derived the strong lower bound

$$\mathrm{span}^0(X_{n,r}) \ge k_{n,r} \tag{6}$$

for $\mathrm{span}^0(X_{n,r})$, where $k_{n,r}$ is defined as follows:

Definition 1.1 $k_{n,r} := \mathrm{span}(nr\xi_{n-1}) - \binom{r+1}{2}$.

Note that we always have $k_{n,r} \geq d_{n,r} - n + 1$, since $\operatorname{span}(nr\xi_{n-1}) \geq nr - (n-1)$ by standard stability properties of vector bundles. Since $d_{n,r}$ is in general much larger than $n-1$, this shows that the resulting inequality $d_{n,r} - n + 1 \leq \operatorname{span}^0(X_{n,r}) \leq d_{n,r}$ already gives relatively sharp estimates for $\operatorname{span}^0(X_{n,r})$, which of course can be improved by applying cohomology theory to reduce the upper bound.

As was shown in [19], $k_{n,r}$ is in fact a lower bound for $\operatorname{span}(X_{n,r})$ as well, except possibly when n is odd and $r = 2$. This seems to be the most intractable case and is studied in some detail in §2. We also prove Theorem 2.3 in §2, which improves the known lower bound for $\operatorname{span}(X_{n-1,2})$ when n is divisible by 8. This involves using an explicit version of the Hurwitz-Radon multiplications, and an Appendix (§5) is added giving an elegant construction of these multiplications based on ideas of Moreno [27] and Lam-Yiu [23], [24].

Then in §3 we prove that if $2 \leq r \leq \rho(n)$, then

$$\operatorname{span}(X_{n,r}) = \operatorname{span}^0(X_{n,r}) = k_{n,r},$$

and we shall calculate $\operatorname{span}(X_{n,r})$ for $r = 2$, 3 or 4, for suitable (infinitely many) values of n, using the lower bound $k_{n,r}$ and other results.

In §4 we prove the following useful inequalities:

$$\begin{aligned} &\operatorname{span}(X_{n,r+1}) \geq \operatorname{span}(X_{n,r}), \\ \text{and, for } s \geq 2, \quad &\operatorname{span}(X_{n,r+s}) \geq \operatorname{span}^0(X_{n,r}) + \binom{s}{2}. \end{aligned} \tag{7}$$

We close §4 with several conjectures about $\operatorname{span}(X_{n,r})$, for which the results in this paper and its predecessors provide strong evidence.

2 On stable span and span of $X_{n,2}$

The projective Stiefel manifold $X_{n,2}$ has an interesting geometric interpretation, as the tangent sphere bundle to P^{n-1}; but this fact will not be used here. Its dimension is of course $2n-3$.

For $\operatorname{span}^0(X_{n,2})$ we have the lower bound given in (6),

$$\operatorname{span}^0(X_{n,2}) \geq k_{n,2} = \operatorname{span}(2n\xi_{n,1}) - 3.$$

In fact, as mentioned in the Introduction, $k_{n,2}$ is known to also be a lower bound for $\operatorname{span}(X_{n,2})$ when n is even. Indeed, the span and stable span of $X_{n,2}$ with n even coincide ([18], [19]), and we shall show, in §3, that $\operatorname{span}(X_{n,2}) = k_{n,2}$ in such cases. Of course, determining $k_{n,r}$ (or in particular $k_{n,2}$) is a special case of the solution of the "generalized vector field problem" (this is the question of what is the span of any multiple of ξ_{n-1} over P^{n-1}, for any n). The latter is not yet completely known, but the results of Lam [21, Theorems 1.1, 3.1, Remark 3.5] and of Lam and Randall [25, 5.14], [26] give the answer in the majority of cases and imply the following proposition. The fact that the binomial coefficient $\binom{2m}{m}$ for $m \geq 1$ is even is implicitly used in those cases where no binomial coefficient is explicitly given; see also §3 for some further details and (in many cases) strengthened results.

Proposition 2.1 *We have the following lower bounds.*

$n = 8m$ & $m \geq 1$ & $\binom{2m}{m-1}$ *odd* $\Rightarrow \operatorname{span}(X_{n,2}) \geq k_{n,2} = n + 5$.
$n = 8m$ & $m \geq 2$ & $\binom{2m}{m-1}$ *even* $\Rightarrow \operatorname{span}(X_{n,2}) \geq k_{n,2} \geq n + 6$,
$\operatorname{span}(X_{16,2}) \geq k_{16,2} = 23$.

$n = 8m+1$ & $m \geq 1 \Rightarrow \operatorname{span}^0(X_{n,2}) \geq k_{n,2} \geq n$,
$\operatorname{span}^0(X_{9,2}) \geq k_{9,2} = 13$, $\operatorname{span}^0(X_{17,2}) \geq k_{17,2} \geq 22$.

$n = 8m+2$ & $m \geq 1 \Rightarrow \operatorname{span}(X_{n,2}) \geq k_{n,2} \geq n + 2$.

$n = 8m+3$ & $m \geq 1 \Rightarrow \operatorname{span}^0(X_{n,2}) \geq k_{n,2} \geq n + 1$.

$n = 8m+4$ & $m \geq 0$ & $\binom{2m+1}{m}$ *odd* $\Rightarrow \operatorname{span}(X_{n,2}) \geq k_{n,2} = n + 1$.
$n = 8m+4$ & $m \geq 2$ & $\binom{2m+1}{m}$ *even* $\Rightarrow \operatorname{span}(X_{n,2}) \geq k_{n,2} \geq n + 2$.

$n = 8m+5$ & $m \geq 0$ & $\binom{2m+1}{m}$ *odd* $\Rightarrow \operatorname{span}^0(X_{n,2}) \geq k_{n,2} = n$.
$n = 8m+5$ & $m \geq 2$ & $\binom{2m+1}{m}$ *even* $\Rightarrow \operatorname{span}^0(X_{n,2}) \geq k_{n,2} \geq n + 1$.

$n = 8m+6$ & $m \geq 0$ & $\binom{2m+1}{m}$ *odd* $\Rightarrow \operatorname{span}(X_{n,2}) \geq k_{n,2} = n - 1$.
$n = 8m+6$ & $m \geq 2$ & $\binom{2m+1}{m}$ *even* $\Rightarrow \operatorname{span}(X_{n,2}) \geq k_{n,2} \geq n + 3$.

$n = 8m+7$ & $m \geq 0$ & $\binom{2m+1}{m}$ *odd* $\Rightarrow \operatorname{span}^0(X_{n,2}) \geq k_{n,2} = n - 2$.
$n = 8m+7$ & $m \geq 2$ & $\binom{2m+1}{m}$ *even* $\Rightarrow \operatorname{span}^0(X_{n,2}) \geq k_{n,2} \geq n + 2$.

□

By a special case of [14, Theorem 1.6], there are one or two isomorphism-classes of $d_{n,2}$-plane bundles over $X_{n,2}$ which are stably isomorphic to the tangent bundle $\tau_{n,2}$; in other words, the James-Thomas number $I(X_{n,2})$ is 1 or 2, respectively. Clearly, $\operatorname{span}^0(X_{n,2}) = \operatorname{span}(X_{n,2})$ if $I(X_{n,2}) = 1$. But, as we shall see in Theorem 2.5, the equality $I(X_{n,2}) = 1$ is a rare phenomenon. In a remark after the proof of Theorem 2.5, we shall outline an idea which *perhaps* can lead to solving the question of whether or not $\operatorname{span}^0(X_{n,2}) = \operatorname{span}(X_{n,2})$ if $I(X_{n,2}) = 2$ and n is odd; but we did not succeed in applying this idea up to now. In view of this, since for n odd Proposition 2.1 gives lower bounds for $\operatorname{span}^0(X_{n,2})$, but not for $\operatorname{span}(X_{n,2})$, the following theorem is useful.

Theorem 2.2 *For the projective Stiefel manifolds $X_{n,2}$ with $n \geq 7$ odd one has* $\operatorname{span}(X_{n,2}) \geq 4$. *In addition to this, one has* $\operatorname{span}(X_{3,2}) = 3$ *and* $\operatorname{span}(X_{5,2}) = 5$.

Proof. Suppose that $n \geq 7$ is odd. Then $d_{n,2} = 2n - 3 \equiv 3 \pmod 4$; since $n \geq 7$, we have $d_{n,2} \geq 11$. It is clear, for instance from the formula (4), that each of the manifolds $X_{n,2}$ is orientable. So by [20, 15.13], in order to prove $\operatorname{span}(X_{n,2}) \geq 4$, it is enough to verify that now $w_2^2(X_{n,2})$ does not vanish, while $w_{d-3}(X_{n,2}) = 0$.

From the formula (4) we obtain $w_2^2(X_{n,2}) = y^4$, and this is not zero (see the description of $H^*(X_{n,r};\mathbb{Z}_2)$ in the Introduction), because $N = n-1 > 4$. In addition to this, $w_{d-3}(X_{n,2}) = \binom{2n}{6} y^{2n-6} = 0$, because we obviously have $2n - 6 > N$. So the theorem is proved for all $n \geq 7$ odd.

Consider the two remaining spaces. Since any orientable 3-dimensional manifold is parallelizable ([34]), span of $X_{3,2}$ is 3. Finally, from Proposition 2.1 we have that $\operatorname{span}^0(X_{5,2}) \geq 5$. But, since $d_{5,2} = 7$, the James-Thomas number of this manifold is 1 (see [14, Theorem 1.7]), and therefore its stable span and span coincide. So we also have $\operatorname{span}(X_{5,2}) \geq 5$. On the other hand, $\operatorname{span}(X_{5,2}) \geq 6$ is impossible; indeed, we have $w_2(X_{5,2}) = y^2 \neq 0$. As a consequence, $\operatorname{span}(X_{5,2}) = 5$. $\square$

Before proving the next theorem, which improves to $\operatorname{span}(S^{n-1})$ the lower bound of 4 (given in Theorem 2.2) for $\operatorname{span}(X_{n-1,2})$ whenever $n \equiv 0 \pmod 8$, some notation needs to be set up. For $r = \rho(n)$, let $e_0, e_1, \ldots, e_{r-1}$ be the canonical orthonormal basis in $\mathbb{R}^r$ and $\varepsilon_0, \ldots, \varepsilon_{n-1}$ be the canonical orthonormal basis in $\mathbb{R}^n$; $\mathbb{R}^{n-1}$ will be the subspace spanned by $\varepsilon_1, \ldots, \varepsilon_{n-1}$. For any vector $c \in \mathbb{R}^n$ we shall write c' for its projection into $\mathbb{R}^{n-1}$. Thus, if $c = (c_0, \ldots, c_{n-1})$, then $c' = (0, c_1, \ldots, c_{n-1}) = c - c_0\varepsilon_0$.

Now let $\mathbb{R}^r \otimes_{\mathbb{R}} \mathbb{R}^n \to \mathbb{R}^n$ be a norm preserving multiplication, denoted $u \otimes v \mapsto u \cdot v := \phi_u(v)$, where $\phi_u \in O(n)$ whenever $\| u \| = 1$. In particular we write $\phi_i(v) = e_i \cdot v$, $0 \leq i \leq r-1$, and by replacing ϕ_i by $\phi_0^{-1} \circ \phi_i$ (which has no effect on the norm preserving property), we may suppose without loss of generality that $\phi_0 = I$. Construction of such norm preserving forms is carried out via [23], [24], [37], and briefly described in the appendix to this

paper. As is shown there, it enjoys the following additional properties, of which (ii) is classical and (iii) describes the first coordinate of $\phi_j(a)$:
P(i) $e_0 \cdot v = v$, i.e. $\phi_0 = I$,
P(ii) for $i, j > 0,\ i \neq j,\quad \phi_i^2 = -I,\quad \phi_i\phi_j + \phi_j\phi_i = 0$,
P(iii) for $j \geq 1$, $a = (a_0, \ldots, a_{n-1}) \in \mathbb{R}^n,\quad e_j \cdot a - (e_j \cdot a)' = \pm a_{J(j)}\varepsilon_0$, where the map $J : \{1, \ldots, r-1\} \rightarrowtail \{1, \ldots, n-1\}$ is injective.

Since, for $i \geq 1$, the orthogonal skew symmetric transformation ϕ_i can be replaced by $-\phi_i$ with no effect on the norm preserving property, we may assume in P(iii) that $e_j \cdot a - (e_j \cdot a)' = -a_{J(j)}\varepsilon_0,\ j \geq 1$.

Theorem 2.3 *One has span*$(X_{n-1,2}) \geq$ *span* (S^{n-1}).

Proof. For $(a, b) \in V_{n-1,2}$, i.e. $a, b \in \mathbb{R}^{n-1}, \| a \| = \| b \| = 1, \langle a, b\rangle = 0$, define $w_j(a, b) = ((e_j \cdot a)', (e_j \cdot b)') \in \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-1},\quad 1 \leq j \leq r-1$.

First we show $w_j(a, b)$ is a tangent vector to $V_{n-1,2}$ at (a, b). We use the explicit description of the tangent bundles to $V_{n,r}$ and $X_{n,r}$ given in [5, Lemma 2.2] (also in [38]). Let $v = (a, b)$ denote a point of the Stiefel manifold $V_{n-1,2}$, and let $[v] = \{v, -v\}$ be the corresponding point in the projective Stiefel manifold $X_{n-1,2}$. Then the tangent space $T_{[v]}(X_{n-1,2})$ consists of pairs $[v, w]$, where $w = (x, y)$ with $x,\ y \in \mathbb{R}^{n-1}$ is such that

$$\langle a, x\rangle = \langle b, y\rangle = \langle a, y\rangle + \langle b, x\rangle = 0, [v, w] = [-v, -w].$$

The tangent space $T_v(V_{n-1,2})$ is similar (without identifications).

Noting that $\langle x, y'\rangle = \langle x, y\rangle$ whenever $x \in \mathbb{R}^{n-1},\ y \in \mathbb{R}^n$, we then have $\langle a, (e_j \cdot a)'\rangle = \langle a, e_j \cdot a\rangle = 0$, similarly for b, and finally

$$\langle a, (e_j \cdot b)'\rangle + \langle b, (e_j \cdot a)'\rangle = \langle a, e_j \cdot b\rangle + \langle b, e_j \cdot a\rangle = \langle a, e_j \cdot b\rangle + \langle -e_j \cdot b, a\rangle = 0.$$

Second, since $w_j(-a, -b) = -w_j(a, b)$, the w_j induce well defined vector fields on $X_{n-1,2}$.

Finally, let us show that $w_1(a, b), ..., w_{r-1}(a, b)$ are linearly independent. So suppose $\sum_{j=1}^{r-1} \lambda_j w_j(a, b) = 0$, with not all λ_j zero. Write $\lambda = \sum_{j=1}^{r-1} \lambda_j \varepsilon_j \in \mathbb{R}^{n-1}$. We also write, for later use, $\widetilde{\lambda} = \sum_{j=1}^{r-1} \lambda_j e_j \in \mathbb{R}^r$. Without loss of generality assume $\| \lambda \| = 1$, so also $\| \widetilde{\lambda} \| = 1$. By definition $\sum_{j=1}^{r-1} \lambda_j (e_j \cdot a)' = \sum_{j=1}^{r-1} \lambda_j (e_j \cdot b)' = 0$. Working with a, and using the equation for c' as well as the property of $e_j \cdot a$ mentioned above, this gives $\sum_{j=1}^{r-1} \lambda_j (e_j \cdot a + a_J \varepsilon_0) = 0$, where we now write $J = J(j)$ for convenience. Thus

$$\sum_{j=1}^{r-1} \lambda_j e_j \cdot a = -(\sum_{j=1}^{r-1} \lambda_j a_J)\varepsilon_0. \tag{8}$$

Trivially, $|\langle \lambda, \sum_{j=1}^{r-1} a_J \varepsilon_j \rangle| = |\sum_{j=1}^{r-1} \lambda_j a_J| = \| -\sum_{j=1}^{r-1}(\lambda_j a_J)\varepsilon_0 \|$. Now, applying successively (8) followed by the norm preserving property, this equals $\| \sum_{j=1}^{r-1}(\lambda_j e_j) \cdot a \| = \| \widetilde{\lambda} \| \, \| a \| = 1 \cdot 1 = 1$. Since the inner product of two vectors, the first being a unit vector and the second having norm at most 1, can have absolute value 1 if and only if they are parallel and the second has norm 1, we have then that $\lambda = \pm \sum_{j=1}^{r-1} a_J \varepsilon_j$. Exactly the same applies to b, so $\lambda = \pm \sum_{j=1}^{r-1} b_J \varepsilon_j$. Finally, since $J = J(j)$ is injective by P(iii), this implies that a, b are also parallel, giving the desired contradiction and completing the proof. □

Corollary 2.4 *We have span*$(X_{7,2}) = 7$.

Proof. The lower bound 7 is obtained from the theorem, and Stiefel-Whitney classes easily give the same upper bound. □

Our next theorem shows that the manifolds $X_{n,2}$ mostly have James-Thomas number 2.

Theorem 2.5 *While the James-Thomas number is* 1 *for* $X_{3,2}$ *and* $X_{5,2}$*, it equals* 2 *for the remaining projective Stiefel manifolds* $X_{n,2}$*, except possibly for* $n = 2^t + 1, t \geq 3$.

Proof. In the proof, we shall suppose $n \neq 2^t+1$. Let BO be the classifying space of the stable orthogonal group O, and let

$$\sigma : H^{i+1}(BO;\mathbb{Z}_2) \longrightarrow H^i(\Omega BO;\mathbb{Z}_2)$$

be the suspension homomorphism. In applying [14, Theorem 1.6], we shall replace the loop space ΩBO by O (see e.g. [2, 2.3.1 (iv)]). Then instead of $\sigma(w_{i+1})$, where w_j is the j-th universal Stiefel-Whitney class, we shall for convenience write $v_i \in H^i(O;\mathbb{Z}_2)$, $i \geq 1$. Now, by [14], it suffices to show that for any map $\beta : X_{n,2} \longrightarrow O$ one has

$$\Delta(\beta) := \beta^*(v_{2n-3}) + \sum_{i=2}^{2n-3} \beta^*(v_{i-1}) w_{2n-2-i}(X_{n,2}) = 0 \in H^{2n-3}(X_{n.2};\mathbb{Z}_2).$$

We have (see (5))

$$H^*(X_{n,2};\mathbb{Z}_2) = \mathbb{Z}_2[y]/(y^N) \otimes V(y_q), \tag{9}$$

as an algebra, where $N = n-1,\ n$ according as n is respectively odd, even, and $q = n-1,\ n-2$ according as n is respectively odd, even (note q is thus always even). Since (4) implies

$$w_{2n-2-i}(X_{n,2}) = \binom{2n}{2n-2-i} y^{2n-2-i},$$

we have

$$\Delta(\beta) = \beta^*(v_{2n-3}) + \sum_{i=2}^{2n-3} \beta^*(v_{i-1}) \binom{2n}{2+i} y^{2n-2-i}.$$

To show this is 0 it will certainly suffice, since $y^{2n-3} = 0$, to prove that $\beta^*(v_j) \in \mathbb{Z}_2[y]/(y^N),\ j \geq 1$. Now we recall that by [9, (8.7)] one has for the Steenrod squares

$$Sq^i(v_j) = \binom{j}{i} v_{i+j}$$

for $i \leq j$. Hence it is sufficient to show that

$$\beta^*(v_{2^k-1}) \in \mathbb{Z}_2[y]/(y^N), \quad k \geq 1\ .$$

This task is trivial if $2^k - 1 < q$, so one only has to consider the range for k where $2^k - 1 \geq q$. Then one has

$$\beta^*(v_{2^k-1}) = \lambda y_q y^{2^k-1-q} \tag{10}$$

for some $\lambda \in \mathbb{Z}_2$. Using [4, 2.1] (see the Introduction), one readily checks that $Sq^1(y_q) = 0$, and applying Sq^1 to (10), we obtain

$$\lambda y_q y^{2^k-q} = \beta^*(v_{2^k}). \tag{11}$$

Observe that the top class in $H^*(X_{n,2};\mathbb{Z}_2)$ is $y_q y^{n-2}$ if n is odd or $y_q y^{n-1}$ if n is even. Hence in all the cases which we need to consider we have $2^k - q < N$; therefore $y_q y^{2^k-q} \neq 0$. On the other hand,

$$\beta^*(v_{2^k}) \in \mathbb{Z}_2[y]/(y^N),$$

because

$$v_{2^k} = Sq^{2^{k-1}} \ldots Sq^2 Sq^1(v_1).$$

Finally, since $2^k \geq q+2 \geq N$, we have $\beta^*(v_{2^k}) = 0$ and (11) implies that $\lambda = 0$. □

We remark that for $X_{2^t+1,2}$ ($t \geq 3$) the problem of determining James-Thomas numbers remains open. To close this section, we outline (as we promised before the statement of Theorem 2.2) a possible way of thinking (in the spirit of [18, p. 8-9]) about the relation between stable span and span of any odd-dimensional smooth closed manifold M with $I(M) = 2$, in particular for $M = X_{n,2}$ when n is odd and $I(X_{n,2}) = 2$. In this case, for any $d_{n,2}$-plane bundle α stably isomorphic to $\tau_{n,2}$ one has (see [33]) a number $b_B(\alpha) \in \mathbb{Z}_2$, called the Browder-Dupont invariant. This b_B distinguishes between those two classes of $d_{n,2}$-plane bundles stably isomorphic to $\tau_{n,2}$, and $b_{\mathrm{B}}(\tau_{n,2})$ is precisely the Kervaire mod 2 semi-characteristic

$$\chi_2(X_{n,2}) = \sum_{i=0}^{n-2} \dim\left(H^i(X_{n,2}; \mathbb{Z}_2)\right) \pmod 2. \tag{12}$$

Observe that for any odd-dimensional $X_{n,r}$, the Kervaire semi-characteristic is nothing but $\frac{1}{2}P_1(X_{n,r}; \mathbb{Z}_2)$ mod 2. From the formula for the Poincaré polynomial $P_t(X_{n,r}; \mathbb{Z}_2)$ it is easy to see that P_1 is divisible by 4 for $r \geq 2$; thus the Kervaire semi-characteristic vanishes in all such cases. Now suppose that we are given some $X_{n,2}$ with n odd and $I(X_{n,2}) = 2$ about which we know that $\mathrm{span}^0(X_{n,2})$ is some number s; then there is a vector bundle η such that $\tau_{n,2} \oplus \varepsilon \approx \eta \oplus (s+1)\varepsilon$. Since, as we have seen, $b_{\mathrm{B}}(\tau_{n,2}) = 0$, it is enough to be able to show that $b_B(\eta \oplus s\varepsilon) = 0$ in order to conclude that $\tau_{n,2} \approx \eta \oplus s\varepsilon$, and $\mathrm{span}(X_{n,2}) = s = \mathrm{span}^0(X_{n,2})$. One can try to proceed analogously knowing that $\mathrm{span}^0(X_{n,2}) \geq k$ for some k (for instance $k = k_{n,2}$), when one wants to show that also $\mathrm{span}(X_{n,2}) \geq k$.

3 The span of $X_{n,r}$ for $r \leq \rho(n)$ and for $r \leq 4$

The following theorem and its corollary allow us to calculate the stable span and also the span of those projective Stiefel manifolds $X_{n,r}$ satisfying $r \leq \rho(n)$, at least to within the knowledge of $k_{n,r}$.

Theorem 3.1 *If $r \leq \rho(n)$, then we have* $\mathrm{span}^0(X_{n,r}) = k_{n,r}$.

Proof. If $r \leq \rho(n)$, then (as shown in the Appendix) there exists a $\mathbb{Z}_2$-equivariant (indeed linear) cross section of the fibre bundle $V_{n,r} \to V_{n,1} = S^{n-1}$. This therefore induces a cross section s of the fibre bundle $\pi : X_{n,r} \to X_{n,1} = P^{n-1}$ such that $s^*(\xi_{n,r}) \approx \xi_{n,1}$. Since also $\pi^*(\xi_{n,1}) \approx \xi_{n,r}$, it follows that $\mathrm{span}(m\xi_{n,r}) = \mathrm{span}(m\xi_{n,1})$ for any m. This yields (see (3) and Definition 1.1)

$$\mathrm{span}^0(X_{n,r}) = \mathrm{span}(nr\xi_{n,r}) - \binom{r+1}{2} = \mathrm{span}(nr\xi_{n,1}) - \binom{r+1}{2} = k_{n,r}.$$

□

Corollary 3.2 *If* $2 \le r \le \rho(n)$ *then we have* $\mathrm{span}(X_{n,r}) = k_{n,r}$.

Proof. The hypotheses imply n is even. Then, as a special case of [19, Theorem, p. 100]), $\mathrm{span}(X_{n,r}) \ge k_{n,r}$. Theorem 3.1 therefore implies now that $\mathrm{span}(X_{n,r}) \ge \mathrm{span}^0(X_{n,r})$, and, as a consequence, $\mathrm{span}(X_{n,r}) = \mathrm{span}^0(X_{n,r})$. □

We next calculate the span for several infinite families of projective Stiefel manifolds $X_{n,r}$ with $2 \le r \le 4$; we shall use various methods for showing, in each case under question, that the lower and upper bounds coincide. For cases (a)-(d), which are strengthenings of results in Proposition 2.1, the following special binomial coefficients are used. All follow readily from Kummer's formula

$$\nu_2\binom{s+t}{t} = \alpha(s) + \alpha(t) - \alpha(s+t),$$

where ν_2 was defined in §1 and $\alpha(t)$ is the number of 1's in the dyadic expansion of t.

$$\nu_2\binom{2m}{m} = \alpha(m) \quad \begin{cases} = 1, & m = 2^a,\ a \ge 0, \\ \ge 2, & \text{otherwise,} \end{cases}$$

$$\nu_2\binom{2m+1}{m} = \alpha(m+1) - 1 \quad \begin{cases} = 0, & m = 2^a - 1,\ a \ge 0, \\ = 1, & m = 2^a + 2^b - 1,\ 0 \le a < b, \\ \ge 2, & \text{otherwise,} \end{cases}$$

$$\nu_2\binom{2m}{m-1} = \alpha(m-1)+\alpha(m+1)-\alpha(m) \quad \begin{cases} = 0, & m = 2^a - 1,\ a > 0, \\ = 1, & m = 2^a + 2^b - 1,\ 0 < a < b, \\ \ge 2, & \text{otherwise.} \end{cases}$$

Theorem 3.3 *We have that* $\operatorname{span}(X_{2t,2}) = k_{2t,2}$, *in particular:*
(a) *If* $n = 8m$, *then* $\operatorname{span}(X_{n,2}) = n + 5$ *for* $m = 2^a - 1$, *i.e.* $n = 2^{a+3} - 8$ $(a > 0)$. *Also* $\operatorname{span}(X_{16,2}) = 23$.
(b) *If* $n = 8m + 2$, *then* $\operatorname{span}(X_{n,2}) = n + 2$ *for* $m = 2^a$, *i.e.* $n = 2^{a+3} + 2$ $(a \geq 0)$.
(c) *If* $n = 8m + 4$, *then* $\operatorname{span}(X_{n,2}) = n + 1$ *for* $m = 2^a - 1$, i.e. $n = 2^{a+3} - 4$ $(a \geq 0)$, *and* $\operatorname{span}(X_{n,2}) = n + 2$ *for* $m = 2^a + 2^b - 1$, *i.e.* $n = 2^{a+3} + 2^{b+3} - 4$ $(0 \leq a < b)$.
(d) *If* $n = 8m + 6$, *then* $\operatorname{span}(X_{n,2}) = n - 1$ *for* $m = 2^a - 1$, *i.e.* $n = 2^{a+3} - 2$ $(a \geq 0)$, *and* $\operatorname{span}(X_{n,2}) = n + 3$ *for* $m = 2^a + 2^b - 1$, *i.e.* $n = 2^{a+3} + 2^{b+3} - 2$ $(0 \leq a < b)$.
In addition, we have:
(e) *For* $m \geq 3$, $\operatorname{span}(X_{2^m-2,3}) = 2^{m+1} - 6$.
(f) *For* $m \geq 2$, $\operatorname{span}(X_{2^m+1,3}) = 2^{m+1} - 3$.
(g) *For* $m \geq 3$, $\operatorname{span}(X_{2^m-2,4}) = 3 \cdot 2^m - 10$.

Proof. The fact that $\operatorname{span}(X_{2t,2}) = k_{2t,2}$ is an immediate consequence of Corollary 3.2, and (a)-(d) are then clear from Proposition 2.1 together with the above formulae for binomial coefficients.

(e) *and* (g) From [21, Theorem 1.1] combined with [19, Theorem, p. 100] (note that for $X_{n,3}$ we could derive a result similar to Proposition 2.1), we obtain that $\operatorname{span}(X_{2^m-2,3}) \geq 2^{m+1} - 6 = k_{2^m-2,3}$. In addition to this, $k_{2^m-2,3}$ is an upper bound, too, because $w_{d-2^{m+1}+6}(X_{2^m-2,3}) = y^{2^m-6}$ does not vanish (note that now $N = 2^m - 4$). This proves (e); part (g) can be proved in an analogous way.

(f) Since $k_{2^m+1,3} = 3 \cdot 2^m + 3 - 2^m - 6 = 2^{m+1} - 3$, we have (applying again [19, Theorem, p. 100]) that $\operatorname{span}(X_{2^m+1,3}) \geq 2^{m+1} - 3$. Now $w_{d-2^{m+1}+3}(X_{2^m+1,3}) = 0$, so more delicate techniques are needed to show that $2^{m+1} - 3$ is also an upper bound for the span. Indeed, in this case both primary and secondary cohomology operations will be used.

We know (see e.g. [18]) that $\operatorname{span}(X_{2^m+1,3}) \geq 2^{m+1} - 2$ would imply the existence of a map $f : X_{2^m+1,3} \to X_{3\cdot 2^m+3,2^{m+1}+4}$ such that $f^*(\xi) \approx \xi$. Hence, in cohomology, we would then have that $f^*(Y) = y$, where

$$H^*(X_{2^m+1,3}; \mathbb{Z}_2) = \mathbb{Z}_2[y]/(y^{2^m}) \otimes V(y_{2^m-2}, y_{2^m})$$

and

$$H^*(X_{3\cdot 2^m+3,2^{m+1}+4}; \mathbb{Z}_2) = \mathbb{Z}_2[Y]/(Y^{2^m}) \otimes V(Y_{2^m}, Y_{2^m+1}, Y_{2^m+2}, \ldots, Y_{3\cdot 2^m+2}).$$

Using the squaring operations as given in §1, the following are easily calculated and will be recorded here for future use in this proof.

$$\begin{aligned} \mathrm{Sq}^1(Y_{2^m}) = 0,\ \mathrm{Sq}^1(y_{2^m}) &= 0, & (13)\\ \mathrm{Sq}^1(y_{2^m-2}) &= yy_{2^m-2}, & (14)\\ \mathrm{Sq}^2(y_{2^m}) &= y^2y_{2^m}, & (15)\\ \mathrm{Sq}^2(Y_{2^m}) &= 0. & (16)\end{aligned}$$

To now show that such a map f cannot exist, we will use the Steenrod algebra $\mathcal{A}_2$ and also the secondary Bockstein cohomology operation β_2 corresponding to the relation $\mathrm{Sq}^1\mathrm{Sq}^1 = 0$. We know (see [11]) that, up to homotopy type, there is a Serre fibration $\pi : X_{n,r} \to P^\infty$, with fibre the Stiefel manifold $V_{n,r}$. Let $i : V_{n,r} \to X_{n,r}$ be fibre inclusion; recall that $H^*(V_{n,r};\mathbb{Z}_2) = V(x_{n-r},\ldots,x_{n-1})$, and we have $i^*(y_j) = x_j$.

If ξ is the Hopf line bundle over P^∞, one has $\pi^*(\xi) \approx \xi$; if we write $H^*(P^\infty;\mathbb{Z}_2) = \mathbb{Z}_2[x]$ with $x = w_1(\xi)$, the equivalent of $\pi^*(\xi) \approx \xi$ is $\pi^*(x) = y$.

In the Serre spectral sequence of the fibration $\pi : X_{2^m+1,3} \to P^\infty$, the element x_{2^m-1} is transgressive with $\tau(x_{2^m-1}) = \mathrm{Sq}^1(x^{2^m-1})$. It follows, by the third Peterson-Stein formula [28, Chap. 16, Theorem 3] and [4], that

$$i^*(\beta_2(\pi^*(x^{2^m-1}))) = \mathrm{Sq}^1(x_{2^m-1}) = x_{2^m},$$

modulo indeterminacy $i^*(\mathrm{Sq}^1(H^{2^m-1}(X_{2^m+1,3};\mathbb{Z}_2)))$. In $H^{2^m-1}(X_{2^m+1,3};\mathbb{Z}_2)$ we take $\{y^{2^m-1},\ yy_{2^m-2}\}$ as basis. Now $\mathrm{Sq}^1(y^{2^m-1}) = y^{2^m} = 0$, also (14) and the Cartan formula imply $\mathrm{Sq}^1(y\cdot y_{2^m-2}) = y^2\cdot y_{2^m-2} + y\cdot yy_{2^m-2} = 0$. Hence the indeterminacy vanishes, and we have

$$i^*\beta_2(y^{2^m-1}) = x_{2^m} \tag{17}$$

as an “honest equation”.

Using the same reasoning in $X_{3\cdot 2^m+3,2^{m+1}+4}$, one finds similarly that

$$i^*\beta_2(Y^{2^m-1}) = x_{2^m}, \tag{18}$$

again with zero indeterminacy.

It follows that $\beta_2(y^{2^m-1}) = a\cdot y^2y_{2^m-2} + b\cdot y_{2^m}$, for some $a, b \in \mathbb{Z}_2$, and $\beta_2(Y^{2^m-1}) = c\cdot Y_{2^m}$, for some $c \in \mathbb{Z}_2$, both with zero indeterminacy. Noting that $i^*(y) = i^*\pi^*(x) = 0$, and using (17), the first equation gives

$$x_{2^m} = i^*\beta_2(y^{2^m-1}) = a\cdot i^*(y^2y_{2^m-2}) + b\cdot i^*(y_{2^m}) = 0 + b\cdot x_{2^m},$$

and therefore $b = 1$. Similarly, using (18), one finds $c = 1$ and thus $\beta_2(Y^{2^m-1}) = Y_{2^m}$.

The naturality of β_2 is expressed by the equation

$$f^*(\beta_2(Y^{2^m-1})) = \beta_2(f^*(Y^{2^m-1})), \tag{19}$$

where the indeterminacy is

$$f^*\mathrm{Sq}^1(H^{2^m-2}(X_{3\cdot 2^m+3,2^{m+1}+4};\mathbb{Z}_2)) + \mathrm{Sq}^1(H^{2^m-2}(X_{2^m+1,3};\mathrm{Z}_2)) = 0,$$

as we have seen above. It follows that

$$f^*(Y_{2^m}) = f^*(\beta_2(Y^{2^m-1})) = \beta_2(f^*(Y^{2^m-1}) = \beta_2(y^{2^m-1}) = a\cdot y^2 y_{2^m-2} + y_{2^m}.$$

We show that $a = 0$. Indeed, we have $0 = f^*(0) = f^*(\mathrm{Sq}^1(Y_{2^m})) = \mathrm{Sq}^1(f^*(Y_{2^m})) = \mathrm{Sq}^1(y_{2^m}) + a\cdot\mathrm{Sq}^1(y^2y_{2^m-2}) = a\cdot y^3y_{2^m-2}$, the last equality following from (13), (14) and the Cartan formula, and thus $a = 0$.

So we have shown

$$f^*(Y_{2^m}) = y_{2^m}.$$

But this implies, using (15), that $\mathrm{Sq}^2(f^*(Y_{2^m})) = \mathrm{Sq}^2(y_{2^m}) = y^2y_{2^m+1}$ does not vanish. On the other hand, using (16), the same element $f^*(\mathrm{Sq}^2(Y_{2^m})) = f^*(0)$ vanishes. Of course, this is a contradiction, and we have shown that $\mathrm{span}(X_{2^m+1,3}) \leq 2^{m+1}-3$ for $m \geq 2$. The proof of part (f), and of the whole Theorem 3.3, is complete. □

We remark that the same techniques used above to compute the secondary Bockstein operation can be used to compute any secondary cohomology operation Φ of degree t on x^{N-t}, $x \in H^1(X_{n,r};\mathbb{Z}_2)$, assuming of course x^{N-t} is in the domain of Φ.

4 Inequalities for the span and some conjectures

A useful piece of information on the span of projective Stiefel manifolds is also the following.

Theorem 4.1 *One has* $\mathrm{span}(X_{n,r+1}) \geq \mathrm{span}(X_{n,r})$*, and, for* $s \geq 2$*,*

$$\mathrm{span}(X_{n,r+s}) \geq \mathrm{span}^0(X_{n,r}) + \binom{s}{2}.$$

Proof. The first assertion is an immediate consequence of the existence of a smooth fibration $p : X_{n,r+1} \to X_{n,r}$. For the second, using the notation of [19], note that vector bundle isomorphisms $\tau_{n,r+1} \approx p^*\tau_{n,r} \oplus \beta'_{n,r+1}$ ((iii) in [19, p. 98]) and $p^*\beta'_{n,r} \approx \beta'_{n,r+1} \oplus \varepsilon$ ((ii) in [19, p. 98]) describe the effect of the pull-back p^* on the tangent bundle $\tau_{n,r}$ and on the twisted orthogonal complement bundle $\beta'_{n,r} \approx \beta_{n,r} \otimes \xi_{n,r}$ ($\beta_{n,r}$ is described in §1). Iterating these isomorphisms s times, one easily establishes inductively, for the fibration $q : X_{n,r+s} \to X_{n,r}$, that

$$\tau_{n,r+s} \approx q^*(\tau_{n,r}) \oplus \binom{s}{2}\varepsilon \oplus s\beta'_{n,r+s}.$$

If $s \geq 2$ then the right hand side can be rewritten

$$q^*(\tau_{n,r} \oplus \varepsilon) \oplus (\binom{s}{2} - 1)\varepsilon \oplus s\beta'_{n,r+s},$$

which has span at least as great as $(\mathrm{span}^0(X_{n,r})+1)+(\binom{s}{2}-1) = \mathrm{span}^0(X_{n,r})+\binom{s}{2}$, completing the proof. □

By saying that r is in the lower range of n we roughly mean that $r < n/2$; see [7] for precise information on the lower range. Based on the results of this paper, [17], and other predecessors we make the following conjectures.

Conjectures 4.2 (A) $\mathrm{span}(X_{n,r}) \geq k_{n,r}$.

(B) $\mathrm{span}(X_{n,r}) = \mathrm{span}^0(X_{n,r})$.

(C) *In the lower range,* $\mathrm{span}(X_{n,r}) = k_{n,r}$.

Remarks 4.3 **(1)** Conjecture 4.2(A) is proved in [19] for all n, r except n odd, $r = 2$.

(2) Conjecture 4.2(B) implies Conjecture 4.2(A), and (B) is proved for roughly 70% of all (n, r) pairs using the results in [20, Ch. 20]; see also [18], [19].

(3) Conjecture 4.2(C) is supported by various results in the present paper, especially Corollary 3.2 and Theorem 3.3, and all other calculations to date. For $n \leq 18$ a small number of exceptions can and do occur, because the product rn can be divisible by $\phi(n-1)$ when $n \leq 18$; here as usual $\phi(n)$ is the number of integers j satisfying $1 \leq j \leq n$ and $j \equiv 0, 1, 2, 4 \pmod 8$. In the upper range, it is usually the case that $\operatorname{span}(X_{n,r}) > k_{n,r}$.

(4) All Conjectures 4.2(A),(B),(C), are true when $r = 1$. This is trivial for (A), (C). For (B), when n is odd, it follows because the Euler characteristic $\chi(X_{n,1}) = \chi(\mathbb{R}P^{n-1}) = 1$ is odd. For n even it is proved in [14].

5 Appendix, Hurwitz-Radon Multiplications

In this appendix a construction of the Hurwitz-Radon multiplications

$$F : \mathbb{R}^r \otimes \mathbb{R}^n \to \mathbb{R}^n, \quad r = \rho(n),$$

is briefly outlined. As in §1, we write $n = (2a+1)2^{c+4d}$, $a, d \geq 0$, $0 \leq c \leq 3$, and $\rho(n) = 2^c + 8d$. The method is that of Lam and Yiu [24], [23], uses Cayley-Dickson algebras (cf. Moreno [27]), and could be considered a shorter and more elegant version of [37]. The facts essential to the proof of Theorem 2.3 will also be established. These multiplications all have the norm-preserving property $\| F(u \otimes v) \| = \| u \| \cdot \| v \|$.

For details of the construction of the Cayley-Dickson algebra $\mathbb{A}_n$, of real dimension 2^n, we refer to [27]. To commence it suffices to recall that $\mathbb{A}_0 = \mathbb{R}$, $\mathbb{A}_1 = \mathbb{C}$, $\mathbb{A}_2 = \mathbb{H}$, $\mathbb{A}_3 = \mathbb{O}$ (respectively the reals, complex numbers, quaternions, and octonions), well known algebras with norm-preserving multiplications. The sedenions $\mathbb{A}_4$ will be discussed and applied in the following paragraph. The algebras $\mathbb{A}_i$, $0 \leq i \leq 3$, with multiplication denoted F, suffice to construct the Hurwitz- Radon multiplications for the case $d = 0$, i.e. $n = (2a+1)2^c$, $0 \leq c \leq 3$ (note that then $r = \rho(n) = 2^c$) by means of the composition

$$\mathbb{R}^r \otimes \mathbb{R}^n = \mathbb{R}^r \otimes (\mathbb{R}^r \otimes \mathbb{R}^{2a+1}) \approx (\mathbb{R}^r \otimes \mathbb{R}^r) \otimes \mathbb{R}^{2a+1} \xrightarrow{F \otimes id} \mathbb{R}^r \otimes \mathbb{R}^{2a+1} = \mathbb{R}^n .$$

The corresponding orthogonal transformations $\phi_0, \phi_1, \ldots, \phi_{r-1}$ (defined in §2) are well known to satisfy properties P(i), P(ii) stated in §2. They are also clearly given by signed permutation matrices with an equal number of plus and minus signs (apart from $\phi_0 = I_n$), since this is true for the multiplication F in $\mathbb{A}_i$. In addition, for $1 \leq i < j \leq r-1$ let us write $v = (x_1, ..., x_n) \in$

$\mathbb{R}^n$, and $\phi_i(v) = (\pm x_{\sigma(1)}, ..., \pm x_{\sigma(n)})$, $\phi_j(v) = (\pm x_{\tau(1)}, ..., \pm x_{\tau(n)})$ for some permutations σ, τ, and some choice of signs. Since $\phi_i(v) \perp \phi_j(v)$, it is clear that $\sigma(k) \neq \tau(k)$, $1 \leq k \leq n$, giving (by taking $k = 1$) property P(iii). This completes the case $d = 0$.

The Lam-Yiu construction, for $d > 0$, gives an inductive procedure such that replacing n by $16n$ will increase r to $r + 8$, this being precisely what the Hurwitz-Radon formula asserts. To this end we turn to the sedenions $\mathbb{A}_4$. According to the Cayley-Dickson construction they consist of ordered pairs (u, v), where $u, v \in \mathbb{O}$, added coordinate-wise and multiplied by the rule $(u, v)(x, y) = (ux - \overline{y}v, yu + v\overline{x})$. Unlike the norm-preserving multiplication in $\mathbb{A}_j$, $j = 0, 1, 2, 3$, $\mathbb{A}_4$ has divisors of zero. However, restricting the multiplication of $\mathbb{A}_4$ to $\mathbb{R}^9 \otimes \mathbb{R}^{16} \to \mathbb{R}^{16}$, where $\mathbb{R}^9 = \mathbb{R}^{\rho(16)}$ is taken to be the subspace

$$\{(a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, 0, 0, 0, 0, 0, 0, 0); a_i \in \mathbb{R}, i = 0, \ldots, 8\},$$

it is easy to show that this restricted multiplication is norm-preserving. We denote it Φ, and it can also be found written in tabular form in [13], p. 4. Taking the standard basis $e_0, e_1, ..., e_8$ for $\mathbb{R}^9$, one defines orthogonal transformations $\gamma_i \in O(16)$, $0 \leq i \leq 8$, by $\gamma_i(v) = \Phi(e_i \otimes v)$. As usual $\gamma_0 = I$, and for $1 \leq i, j \leq 8$, γ_i is skew symmetric, $\gamma_i^2 = -I$, and $\gamma_i \gamma_j + \gamma_j \gamma_i = 0$, $i \neq j$, it is also standard that these properties are equivalent to the multiplication being norm-preserving. In addition it is clear, using the definition or the table in [13], that each γ_i, $i > 0$, is a signed permutation matrix with an equal number of plus and minus signs.

To effect the Lam-Yiu construction, one also defines $\Gamma = \gamma_1 \cdots \gamma_8$. One can easily verify that Γ is symmetric, $\Gamma^2 = I$, and for $i > 0$ one has $\Gamma \gamma_i + \gamma_i \Gamma = 0$. Being a product of signed permutation matrices it must also be a signed permutation matrix, indeed, it is easy to check that for $x = (x_1, ..., x_{16})$, $\Gamma(x) = (-x_9, x_{10}, x_{11}, ..., x_{16}, -x_1, x_2, x_3, ..., x_8)$.

Now suppose one has a norm-preserving multiplication $F : \mathbb{R}^r \otimes \mathbb{R}^n \to \mathbb{R}^n$, then just as for Φ one defines $r = \rho(n)$ orthogonal transformations $\theta_0, ..., \theta_{r-1} \in O(n)$. As in §2 (just before P(i)-P(iii)), we may suppose that $\theta_0 = I$. Then the remaining θ_i, $1 \leq i \leq r - 1$, are skew symmetric and satisfy the same identities as the γ_i above. In addition, we assume (inductively) that they are signed permutation matrices. Then we construct a norm-preserving multiplication $G : \mathbb{R}^{\rho(n)+8} \otimes \mathbb{R}^{16n} \to \mathbb{R}^{16n} \approx \mathbb{R}^n \otimes \mathbb{R}^{16}$ by defining the corresponding $\rho(n)+8=\rho(16n)$ orthogonal transformations in $O(16n)$ as

$$\phi_0 = I, \quad \phi_1 = \theta_1 \otimes \Gamma, \ \ldots, \phi_{\rho(n)-1} = \theta_{\rho(n)-1} \otimes \Gamma,$$
$$\phi_{\rho(n)} = I \otimes \gamma_1, \ \ldots, \phi_{\rho(n)+7} = I \otimes \gamma_8.$$

One easily checks that the ϕ_i satisfy the same identities as the γ_i, θ_i, and hence define a norm-preserving multiplication on $\mathbb{R}^{16n}$. Furthermore, the tensor product of signed permutation matrices is obviously again a signed permutation matrix. Starting from the already completed case $d = 0$, this construction inductively produces the Hurwitz-Radon multiplications, as a family of skew symmetric signed permutation matrices (and the first being I). This establishes properties P(i), P(ii) used in the proof of Theorem 2.3, and an argument similar to that used above in the $d = 0$ case, also gives property P(iii).

The fact established above, that the Hurwitz-Radon multiplications are given by signed permutation matrices, is also of interest in the combinatorial study of Hadamard matrices. Although this fact is likely known, perhaps even since the time of Hurwitz and Radon, to the best of the authors' knowledge it (or its proof) does not appear in the literature.

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Július Korbaš
Department of Algebra, Geometry, and Mathematical Education,
Faculty of Mathematics, Physics, and Informatics,
Comenius University, Mlynská dolina, SK-842 48 Bratislava 4, Slovakia
or
Mathematical Institute, Slovak Academy of Sciences,
Štefánikova 49, SK-814 73 Bratislava 1, Slovakia
(korbas@fmph.uniba.sk)
and
Peter Zvengrowski
Department of Mathematics and Statistics,
The University of Calgary, Calgary,
Alberta T2N 1N4, Canada
(zvengrow@ucalgary.ca)

References

[1] J. F. Adams, *Vector fields on spheres*, Ann. of Math. **75** (1962), 603-632.

[2] J. F. Adams, Infinite Loop Spaces, Princeton Univ. Press, Princeton, N. J. 1978.

[3] D. Ajayi, S. Ilori, *Stiefel-Whitney classes of the flag manifold* $\mathbb{R}F(1,1,n-2)$, Czechoslovak Math. J. **52** (2002), 17-21.

[4] E. Antoniano, *La accion del algebra de Steenrod sobre las variedades de Stiefel proyectivas*, Bol. Soc. Mat. Mexicana **22** (1977), 41-47.

[5] E. Antoniano, S. Gitler, J. Ucci, P. Zvengrowski, *On the K-theory and parallelizability of projective Stiefel manifolds*, Bol. Soc. Mat. Mexicana **31** (1986), 29-46.

[6] N. Barufatti, D. Hacon, *K-theory of projective Stiefel manifolds*, Trans. Amer. Math. Soc. **352** (2000), 3189-3209.

[7] N. Barufatti, D. Hacon, K. Y. Lam, P. Sankaran, P. Zvengrowski, *The order of real line bundles*, Bol. Soc. Mat. Mexicana **10** (2004), 149-158.

[8] P. Baum, W. Browder, *The cohomology of quotients of classical groups*, Topology **3**(1965), 305-336.

[9] A. Borel, *Sur l'homologie et la cohomologie des groupes de Lie compacts connexes*, Amer. J. Math. **76** (1954), 273-342.

[10] G. Bredon, A. Kosinski, *Vector fields on* π*-manifolds*, Ann. of Math. **84** (1966), 86-90.

[11] S. Gitler, D. Handel, *The projective Stiefel manifolds - I*, Topology **7** (1968), 39-46.

[12] S. Ilori, D. Ajayi, *Vector fields on the real flag manifolds* $\mathbb{R}F(1,1,n-2)$, Math. Slovaca **58** (2008), 127-129.

[13] I. M. James, The Topology of Stiefel Manifolds, Cambridge University Press, Cambridge 1976.

[14] I. James, E. Thomas, *An approach to the enumeration problem for non-stable vector bundles*, J. Math. Mech. **14** (1965), 485-506.

[15] J. Korbaš, *On the vector field problem for* $O(n)/O(1)\times O(1)\times O(n-2)$, Acta Math. Hungarica **105** (2004), 123-131.

[16] J. Korbaš, *Distributions, vector distributions, and immersions of manifolds in Euclidean spaces*, Chapter 13 (pp. 665-724) in: Handbook of Global Analysis (edited by D. Krupka and D. Saunders), Elsevier, Amsterdam 2008.

[17] J. Korbaš, P. Sankaran, P. Zvengrowski, *Span of projective Stiefel manifolds*, Preprint, Univ. of Calgary Yellow Series 704 (1991).

[18] J. Korbaš, P. Zvengrowski, *The vector field problem: a survey with emphasis on specific manifolds*, Exposition. Math. **12** (1994), 3-30.

[19] J. Korbaš, P. Zvengrowski, *On sectioning tangent bundles and other vector bundles*, Proc. Winter School Geometry and Physics, Srní (Czech Republic) 1994, Rend. Circ. Mat. Palermo (II), Supplemento **39** (1996), 85-104.

[20] U. Koschorke, Vector Fields and Other Vector Bundle Morphisms - A Singularity Approach, Lecture Notes in Math. 847, Springer-Verlag, Berlin 1981.

[21] K. Y. Lam, *Sectioning vector bundles over real projective space*, Quart. J. Math. Oxford (2) **23** (1972), 97-106.

[22] K. Y. Lam, *A formula for the tangent bundle of flag manifolds and related manifolds*, Trans. Amer. Math. Soc. **213** (1975), 305-314.

[23] K. Y. Lam, *Topological methods in the study of bilinear forms*, Perspectives in Geometry and Topology, Proc. of the Ind. Inst. Tech, Bombay Golden Jubilee International Workshop/Conference in Geometry and Topology, to appear.

[24] K. Y. Lam, P. Y. H. Yiu, *Sums of squares formulae near the Hurwitz-Radon range*, Contemp. Math. **58** part II (1987), 51-56.

[25] K. Y. Lam, D. Randall, *Geometric dimension of bundles on real projective spaces.* Contemp. Math. **188** (1995), 137-160.

[26] K. Y. Lam, D. Randall, *Periodicity of Geometric Dimension for Real Projective Spaces*, Progress in Mathematics, Vol. 136 (1996), 223-234.

[27] G. Moreno, *The zero-divisors of the Cayley-Dickson algebras over the real numbers*, Bol. Soc. Mat. Mex. **4** (1998), 13-27.

[28] R. Mosher, M. Tangora, Cohomology Operations and Applications in Homotopy Theory, Harper & Row, Publishers, New York 1968.

[29] O. Saeki, *Notes on the topology of folds*, J. Math. Soc. Japan **44** (1992), 551-566.

[30] O. Saeki, *Fold maps on* 4*-manifolds*, Comment. Math. Helvetici **78** (2003), 627-647.

[31] P. Sankaran, P. Zvengrowski, *Upper bounds for the span of projective Stiefel manifolds*, Recent developments in algebraic topology, 173-181, Contemp. Math., **407**, Amer. Math. Soc., Providence, RI, 2006.

[32] L. Smith, *Some remarks on projective Stiefel manifolds, immersions of projective spaces and spheres*, Proc. Amer. Math. Soc. **80** (1980), 663-669.

[33] W. Sutherland, *The Browder-Dupont invariant*, Proc. London Math. Soc. (3) **33** (1976), 94-112.

[34] E. Thomas, *Vector fields on manifolds*, Bull. Amer. Math. Soc. **75** (1969), 643-683.

[35] K. Yamaguchi, *Spaces of free loops on real projective spaces*, Kyushu J. Math. **59** (2005), 145-153.

[36] K. Yamaguchi, *The homotopy of spaces of maps between real projective spaces*, J. Math. Soc. Japan **58** (2006), 1163-1184.

[37] P. Zvengrowski, *Canonical vector fields on spheres*, Comment. Math. Helvetici **43** (1968), 341-347.

[38] P. Zvengrowski, *Über die Parallelisierbarkeit von Stiefel Mannigfaltigkeiten*, Preprint, Forschungsinst. für Math. Zürich (1976).

[39] P. Zvengrowski, *Remarks on the span of* $X_{n,r}$, XI Brazilian Topology Meeting (Rio Claro 1998), 85-98, World Sci. Publishing, River Edge, NJ 2000.